

# Knowing a network by walking on it: emergence of scaling

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A model for growing networks is introduced, having as a main ingredient that new nodes are attached to the network through one existing node and then explore the network through the links of the visited nodes. From exact calculations of two limiting cases and numerical simulations the phase diagram of the model is obtained. In the stationary limit, large network sizes, a phase transition from a network with finite average connectivity to a network with a power law distribution of connectivities, with no finite average, is found. Results are compared with measurements on real networks.

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A network is composed by a set of nodes and a set of links among them. The topological properties of disordered networks have been studied for a long time. A well known example is the work of Erdos and Rényi [1] where a network generated by placing links among the nodes at random is studied. However such a model is not able to describe the topological properties of real complex networks. The main current studies are thus focused in finding the mechanism which generates such networks [2–9].

Watts and Strogatz [2] introduced the "small-world" network model, an interpolation between regular lattices and random graphs. Different social, biological and economic networks has been found to be well described by such approach [2,3]. Such a model is more appropriate for networks where the number of nodes remains constant. Moreover, it yields a distribution of connectivities  $P(k)$  peaked around a characteristic value [9].

On the other hand, some authors have studied the topological properties of networks generated by evolutionary dynamics [6,7]. In this case the network topology is changed using extremal dynamics rules inspired in the Bak-Sneppen model for biological evolution [10]. A model with fixed [6] and variable [7] number of nodes has been proposed. The study of the topological properties of the second model reveals that the distribution of connectivities changes in time, yielding either exponential or power law distributions.

Finally, there is a class of growing-network models where the addition of new nodes leads to scale-free structures [8,9]. In this case the connectivity distribution follows a power law decay  $P(k) \sim k^{-\gamma}$  ( $2 \leq \gamma \leq 3$ ). Examples are the World Wide Web (WWW) where HTML documents are the nodes and the links to other documents in the WWW are the links [9,11]; and the citation of scientific publications where papers are the node and citations among them are the links [12]. In these two examples the number of nodes (HTML documents or pa-

pers) is clearly increasing in time. Here the attention is focused in this class of networks.

Different points of view appear when describing the evolution in time of the set of links, which is actually the mechanism introducing randomness in scale-free models. In the approach by Huberman and Adamic [8] the number of links pointing to a node is a random fraction of the number of links which are already pointing to that node. On the other hand, Barabási and Albert [9] have proposed a preferential attachment, where new nodes are linked with higher probability to those existing nodes with have higher connectivity. This model has been recently shown to be a particular case of a model proposed by Simon in the fifties [13]. The study of this class of growing networks is currently very active and new variants have been proposed [14–16], keeping the preferential attachment as main ingredient.

However, these growing-network models do not take into account one fundamental property of real networks, the fact that a new node does not have "knowledge" of the entire network. For instance, when a scientist is writing a manuscript he does not know all the already published papers which may have certain relation with the subject he is dealing with. In fact he only knows a few number of papers and through the references appearing on them he found new ones, and continues his search recursively using the new references on them. Thus, the model introduced in this paper is based on the fact that we know the network, or at least part of it, by "walking" on it. This feature together with growing yield the scaling behavior observed in real growing networks.

The model is defined by giving an initial condition and a set of evolution rules. *Initial condition*: one starts with one node  $N = 1$  and an empty set of links. The evolution rules are divided in *adding* a node and *walking* through the network. *Adding*: A new node  $N + 1$  is created with a link to one existing node selected at random. *Walking*: the new node "walks" through all the nodes pointed by

the selected node and create a link to them with a probability  $p$ . This last rule is repeated recursively with the new selected links. When no new link is created add a new node.

One run of this algorithm for  $p = 0.5$  and up to  $N = 5$  nodes is shown in Fig. 1.  $N = 1$ : a node is created (node 1).  $N = 2$ : a second node (node 2) is created which can only point to node 1.  $N = 3$ : node 3 is created and it can point either to node 1 or 2. In the particular case shown in Fig. 1 it points to node 1. Since node 1 does not have any link the rule stops.  $N = 4$ : node 4 is created which can point to either node 1,2 and 3. In this case it points to node 2. Now node 2 has a link to node 1 so with probability  $p$  node 4 creates a link to node 1 (it is not created in this case).  $N = 5$ : node 5 is created which can point to either node 1,2,3 and 4. In this case it points to node 4. Since 4 has a link to node 2 node 5 will create a link to node 2 with probability  $p$  (it is created in this case). But now node 2 has a link to node 1 so with probability  $p$  node 5 creates a link to node 1 (it is not created in this case). And so on.

The main assumptions of this model is that one has the first contact with the network through one node and then explores the rest of by "walking" through the directed links. Moreover, there is a time scale separation between the addition of nodes and the mechanism of creation of new links. The network is clearly a directed graph and between two nodes there can be only one link, which goes from one to the other. The only parameter of the model is  $p$  which may have different interpretations according to the particular problem one is modeling. For instance, in the problem of citations  $p$  is the fraction of papers appearing in the list of references of one paper which may be of our interest.

Let us now investigate the evolution of the connectivity distribution as  $N$  grows. Here the connectivity is defined as the number of links pointing to a node (in-degree). When a new node is added to the network the connectivity of any node already at the network remains constant or increases by one. For instance, in Fig. 1, from  $N = 3$  to 4 the connectivity of node 2 increases by one while that of the other nodes remain constant. Moreover, the created node has connectivity  $k = 0$

Let  $w(k, N)$  be the probability that when adding the  $N + 1$  node the connectivity of a node with connectivity  $k$  increases by one. With this definition, the number of nodes  $n(k, N)$  with connectivity  $k$  evolves according to the set of equations

$$n(0, N + 1) = n(0, N) + 1 - w(0, N)n(0, N), \quad (1)$$

$$n(k, N + 1) = n(k, N) + w(k - 1, N)n(k - 1, N) - w(k, N)n(k, N), \quad \text{for } k > 0. \quad (2)$$

For  $N \gg 1$  one can look for stationary solutions of this set equations. In this limit  $w(k, N)$  should be of the form

$w(k, N) = W(k)/N$ , where  $1/N$  comes from the fact that the new node is attached to an existing node selected at random, which happens with probability  $1/N$  (this will be demonstrated below for two limiting cases). Then, taking into account that  $n(k, N) = NP(k, N)$  and the stationary condition  $P(k, N + 1) = P(k, N) = P(k)$  one obtains

$$P(0) = W(0)/2, \quad (3)$$

$$P(k) = W(k - 1)/[1 + W(k)], \quad \text{for } k > 0. \quad (4)$$

Thus, determining  $W(k)$  one can iterate (4) to obtain the stationary distribution  $P(k)$ .

A node with connectivity  $k = 0$  can only increase its connectivity if the new node is attached to it, which happens with probability  $1/N$ . Hence,  $w(0, N) = 1/N$  and, therefore,  $W(0) = 1$ . From this result and (3) it follows that  $P(0) = 1/2$ . This result is independent of the value of  $p$ , i.e. in the present model half of the nodes have no links pointing to them.

The form of  $w(k, N)$  for  $k > 0$  is not known. Here only the limiting cases  $p = 0$  and  $p = 1$  are solved exactly. For  $p = 0$ , independent of the connectivity of a node, the probability that its connectivity increases by one is  $w(k, N) = 1/N$ , which is just the probability that the new node is attached to it. Hence,  $W(k) = 1$  independent of  $k$ . Substituting this result in (4) and iterating with the initial condition  $P(0) = 1/2$  it results that

$$P(k) = 2^{-(k+1)}, \quad \text{for } p = 0. \quad (5)$$

In the other limit,  $p = 1$ , a node will increase its connectivity either if the new node is attached to it or to one of the nodes with a link to it, i.e.  $w(k, N) = (1 + k)/N$ . In this case after iteration of (4) one obtains

$$P(k) = [(k + 1)(k + 2)]^{-1}, \quad \text{for } p = 1. \quad (6)$$

Notice that for  $p = 1$  since  $w(k, N) = (1 + k)/N$  there is a preferential attachment to nodes with larger connectivity. This is one of the main ingredients introduced by Barabási and Albert to obtain the desired emergence of scaling [9]. Actually in their model  $w(k, N)$  is also linear in  $k/N$ . However, the preferential attachment is imposed while here, on the contrary, it appears self-consistently from the dynamics of the network. The evolution rules of the model do not show us *a priori* the existence of a preferential attachment but it is clear that nodes with larger connectivity becomes more visible when one "walks" through the network.

These limiting cases are described by distributions which are qualitatively different. For  $p = 0$  the distribution is exponential with a finite average connectivity. On the contrary for  $p = 1$  the distribution is the power law decay  $P(k) \sim k^{-2}$  for large  $k$ . This power law decay goes

up to the largest possible connectivity  $k = N - 1$  while  $P(k, N) = 0$  for  $k \geq N$ . Then for large  $N$  the average connectivity scale as

$$\langle k \rangle = A + \ln N, \quad (7)$$

where  $A$  is independent of  $N$ . The average connectivity thus diverges when  $N \rightarrow \infty$ .

Since the limiting cases  $p = 0$  and  $p = 1$  give qualitative different behaviors there should be certain probability threshold  $p_c$  where a transition from a network with finite average connectivity to a free-scale network takes place. In the absence of analytical results for  $0 < p < 1$  numerical simulations are performed in order to explore this part of the phase diagram.

The maximum network size reached was 81920 nodes and average was taken over 100 runs of the algorithm which generates the network, for each value of  $p$  reported here. The resulting connectivity distribution is shown in Fig. 2. The first thing to be notice is that the analytical results in (5) and (6) for the limiting cases  $p = 0$  and  $p = 1$ , respectively, are in very good agreement with the numerical data. Second, the transition from a finite average distribution to a power law takes place at an intermediate probability  $0 < p_c < 1$ , where  $p_c$  is in the neighborhood of 0.4.

To obtain a more precise estimate of the threshold the scaling of the average connectivity  $\langle k \rangle$  with  $N$  was investigated. For  $p < p_c$  it was found to saturate to a finite value when  $N \gg 1$  while for  $p > p_c$  it grows logarithmically with  $N$  as in the limiting case  $p = 1$ . The results for  $p = 0.1$  and  $p = 0.9$  are shown in Fig. 3. In the neighborhood of the threshold the network sizes needed to reach the stationary state are not accessible by the present numerical results. Thus, the following approximate method was used to determine  $p_c$ .

The numerical data was fitted by the parabola  $\langle k \rangle(N) = a + bx + cx^2$ , where  $x = \log N$  and  $a$ ,  $b$  and  $c$  are fitting parameters. For  $p > p_c$  the parameter  $c$  is expected to be zero while for  $p < p_c$  it is negative, as a consequence of the tendency of  $\langle k \rangle(N)$  to saturates to the stationary value. Actually due to numerical errors  $c$  will never be exactly zero. Here  $p_c$  is estimated by the value of  $p$  at which  $c$  changes sign, becoming either zero or positive. Using this criteria it results that  $p_c = 0.39 \pm 0.01$ .

Let us now focus our attention in the form of  $P(k)$ . In Fig. 2 it can be seen that for  $p < p_c$  the shape of the connectivity distribution depends on  $p$ . For  $p > p_c$ , although there are some deviations for  $k$  small, the large  $k$  behavior is characterized by the power law decay  $P(k) \sim k^{-\gamma}$  with an exponent  $\gamma = 2.0 \pm 0.1$ . Thus, above the threshold, the connectivity distribution of the network is very robust, showing little variations when  $p$  changes.

These features are very similar to those observed in some sandpile models [17,18], the paradigm of the theory of self-organized criticality [19]. As in these models,

there is a time scale separation, here between the addition of new nodes and their "walk" through the network. In the thermodynamic limit, large system sizes, the phase diagram of the model is divided in a sub-critical and a critical region, and in the critical region the power law exponent does not depend on the control parameter. All these similarities put these models in the same class, with a self-organized critical region in the phase diagram.

There are real complex networks which can be described by the present model. The network of citations among papers published in journals is an example. In this case  $k$  is the number of times a paper is cited in other papers. The analysis of the available data yield the power law exponent  $\gamma = 3$  [12]. This value is larger than the universal value  $\gamma = 2$  observed in the critical region of the model introduced here. Thus, it seems that citation problem is in some part of the sub-critical region below  $p_c$ . Actually the fraction of papers one usually considers, to be referred in a future publication, is small in comparison with the total amount of referenced papers appearing in papers of our knowledge. However, more data is needed to reach to a final conclusion.

In the WWW network HTML documents are the nodes, the links to other documents in the WWW are the links, and  $k$  is the number of times a HTML document appears as a link in other HTML documents. In HTML documents the links are created when the HTML document is created but can also be changed latter on. Hence, the addition of new nodes is not the only mechanism of changing the set of links. However, the rate of addition of new HTML documents is actually very high so one expect that the addition of new documents is the dominant mechanism and, therefore, can be described by the present model. Measurements reported in the literature [9,11] yield the exponent  $\gamma = 2.1 \pm 0.1$  in very good agreement with the exponent  $\gamma = 2$  in the critical region. Hence, the WWW network is in some part of the critical region.

Thus, with only one control parameter the present model is able to describe the form of the connectivity distribution of networks with different topologies. For  $0 \leq p < p_c$  it describes networks with a finite average connectivity, which may have a power law decay for small connectivities but with a cutoff independent of the network size for large connectivities. On the contrary, for  $p_c < p \leq 1$  it describes networks with power law distribution of connectivities, up to a cutoff determined by the network size. The transition from one behavior to the other is determined by the parameter  $p$ , which measures the probability to create a new link and continue the search in the network through the added link.

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- [1] B. Bollobás, *random Graphes* (Academic Press, london, 1985).
- [2] D. J. Watts and S. H. Strogatz, *Nature* **393**, 440 (1998); for a recent review see M. E. J. Newman, arXiv:cond-mat/0001118.
- [3] L. A. Nunes Amaral, A. Scala, M. Barthélémy, and H. E. Stanle, arXiv: cond-mat/0001458.
- [4] S. Solomon, G. Weisbuch, L. de Arcangelis, N. Jan, and D. Stauffer, *Physics A* **277**, 239 (2000).
- [5] S. Jain and S. Krishna, *Phys. Rev. Lett.* **81**, 5684 (1998).
- [6] K. Christensen, R. Donangelo, B. Koller, and K. Sneppen, *Phys. Rev. Lett.* **81**, 2380 (1998).
- [7] F. Slanina and M. Kotrla, *Phys. Rev. Lett.* **83**, 5587 (1999); arXiv:cond-mat/0004407.
- [8] B. A. Huberman and L. A. Adami, *Nature* **401**, 131 (1999).
- [9] A.-L. Barabási and R. Albert, *Science* **286**, 50 (1999); A.-L. Barabási, R. Albert, and H. Jeong, *Physica A*, **272**, 173 (1999).
- [10] P. Bak and K. Sneppen, *Phys. Rev. Lett.* **71**, 4083 (1993).
- [11] A. Broder *et al*, *Econophysics Forum* preprint, June (2000), <http://www.unifr.ch/econophysics/>.
- [12] S. Redner, *Eur. Phys. J. B* **4**, 131 (1998).
- [13] S. Bornholdt and H. Ebel, cond-mat/0008465.
- [14] S. N. Dorogovtsev, J. F. F. Mendes, and A. N. Samukhin, arXiv:cond-mat/0004434; S. N. Dorogotsev and J. F.F. Mendes, arXiv:cond-mat/0005050.
- [15] R. Albert and A.-L. Barabási, arXiv:cond-mat/0005085.
- [16] P. L. Krapivsky, S. Redner, and F. Leyvraz, arXiv:cond-mat/0005139.
- [17] B. Tadić and D. Dhar, *Phys. Rev. Lett.* **79**, 1519 (1997).
- [18] A. Vázquez and O. Sotolongo-Costa, *J. Phys. A* **32**, 2633 (2000).
- [19] P. Bak, C. Tang, and K. Wiesenfeld, *Phys. Rev. Lett.* **59**, 381 (1987); *Phys. Rev. A* **38**, 364 (1988).

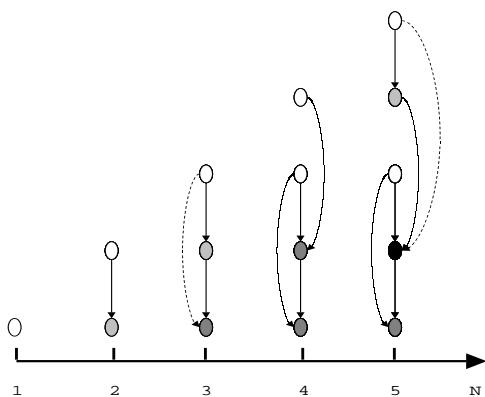


FIG. 1. One run up to 5 nodes of the algorithm with generates the network using  $p = 0.5$ . The number of nodes in the network is indicated in the horizontal axis. Different gray levels indicate different connectivities from  $k = 0$  (white) to  $k = 3$  (black). Dashes lines indicates that the new node performed one "walk" before creating this link.

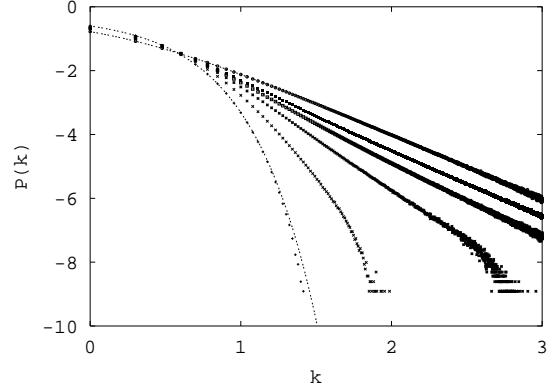


FIG. 2. Connectivity distribution for different values of  $p$ . The points corresponds to, from left to right,  $p = 0$ ,  $p = 0.1$ ,  $p = 0.2$ ,  $p = 0.3$ ,  $p = 0.4$ , and  $p = 1$ , respectively. The continuous lines corresponds with the exact results for  $p = 0$  and  $p = 1$  in Eqs. (5) and (6), respectively.

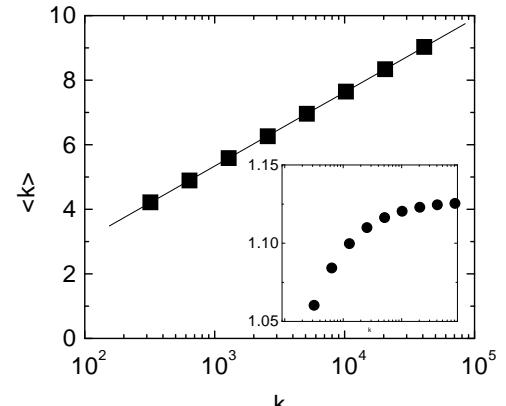


FIG. 3. Average connectivity as a function of the number of nodes  $N$  added to the network for  $p = 0.1$  (inset) and  $p = 0.9$  (full plot), in a semi-log scale. For  $p = 0.1$  the plot clearly saturates to a finite value. On the contrary, for  $p = 0.9$   $\langle k \rangle$  grows logarithmically, which is manifested as a straight line in the semi-log plot.